Smith **Math 290-A** **Fall 2009**

Final Exam **December** 16, 2009

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do two (2) of these "Computational" problems

C.1. [15 points] Solve the following system of equations **by hand.**

$$\begin{cases} x_3 - x_4 - x_5 = 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \end{cases}$$

C.2. [15 points] Let V be the vector space of all functions of the form $f(t) = c_1 \cos(t) + c_2 \sin(t)$ where $c_1$and c_2 are arbitrary complex numbers. That is, V is the subspace of F with basis $B = \{\cos(t), \sin(t)\}$.

1. Find the matrix representation $M_{B,B}^{T}$ of the linear transformation $T : V \to V$ defined by $T(f) = f'' + 3f' + 2f$.
2. Is T an isomorphism?

C.3. [15 Points] Find a basis for the range of the linear transformation $T : M_{2,2} \to M_{2,2}$ defined by $T(A) = \frac{1}{2}A + \frac{1}{2}A^t$.

Do any two (2) of these "Similar to In Class, Text, or Homework" problems

M.1. [15 Points] Prove that $T : U \to V$ is a linear transformation if and only if $T(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) = \alpha_1 T(\vec{u}_1) + \alpha_2 T(\vec{u}_2)$ for all $\vec{u}_1, \vec{u}_2 \in U$ and all $\alpha_1, \alpha_2 \in \mathbf{C}$.

M.2. [15 Points] Prove that if $T : U \to V$ is a linear transformation and X is a subspace of V then the pre-image of X under T, $T^{-1}(X) = \{\vec{u} \in U : T(\vec{u}) \in X\}$, is a subspace of U.

M.3. [15 Points] Use matrix multiplication notation to prove that if $A \in M_{mn}$ and $B \in M_{np}$ then

1. $N(B) \subseteq N(AB)$
2. $C(AB) \subseteq C(A)$

Do two (2) of these "Other" problems

T.1. [15 Points] Given that $A \in M_{mn}$ and $B \in M_{nm}$ where $m \neq n$ and $AB = I_m$. Use a proof by contradiction to show that the columns of B must be linearly independent.

T.2. [15 Points] Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and define a function $T : P_2 \to M_{2,2}$ by $T(p) = p(A)$.

1. Show T is a linear transformation.
2. Determine if T is injective by computing the null space of the matrix representation $M^T_{B,C}$ where B and C are the standard bases of P_2 and M_{22}, respectively.

T.3. [15 Points] Let V be a subspace of $\mathbf{C}^n$ and define the orthogonal complement of V by $V^\perp = \{\vec{x} \in \mathbf{C}^n \mid \langle \vec{x}, \vec{v} \rangle = 0 \text{ for every } \vec{v} \in V\}$.

1. Show that $V^\perp$ is a subspace of $\mathbf{C}^n$.
2. Find a basis of $V^\perp$ in the special case where $V = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} \right\rangle \subseteq \mathbf{C}^3$.

T.4. [15 Points] Let $T : M_{nn} \to M_{nn}$ be defined by $T(A) = A^t$. Find all eigenvalues of T and describe all of the eigenspaces. [Hint: consider the linear transformation $T \circ T$.]

T.5. [15 points] Professor Beezer has proven that if V is a finite-dimensional vector space and $T : V \to V$ has $\ker(T) = \left\{\vec{0}\right\}$ then T is an isomorphism. Show that this is not necessarily the case if V is infinite dimensional by giving an example of a linear transformation $T : P \to P$ that is not surjective but that has $\ker(T) = \left\{\vec{0}\right\}$.[Recall that P is the infinite dimensional vector space of **all** polynomials.]

You must do both of these problems ON THIS SHEET

R.1. [15 points] Prove that the set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{C}^3 : \ -2x_1 + 4x_2 + 3x_3 = 0 \right\}$ is a subspace of $\mathbf{C}^3$ by applying the three-part test of Theorem TSS. Write your proof according to the standards of this semester's writing exercises.

R.2. [15 points] Part of Theorem NPNT ("Nonsingular Products, Nonsingular Terms") says: If A and B are square matrices of the same size, and AB is nonsingular, then B is nonsingular. Construct a proof by contradiction of this fact and write your proof according to the standards of this semester's writing exercises. (You may **not** use Theorem NPNT in your proof.)